

A blow-up criterion for compressible viscous heat-conductive flows

Song Jiang Yaobin Ou

LCP, Institute of Applied Physics and Computational Mathematics,

P. O. Box 8009, Beijing 100088, P.R.China

E-mail: jiang@iapcm.ac.cn, ou.yaobin@gmail.com

Abstract

We study an initial boundary value problem for the Navier-Stokes equations of compressible viscous heat-conductive fluids in a 2-D periodic domain or the unit square domain. We establish a blow-up criterion for the local strong solutions in terms of the gradient of the velocity only, which coincides with the famous Beale-Kato-Majda criterion for ideal incompressible flows.

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1 Introduction

This paper is concerned with blow-up criteria for the two-dimensional Navier-Stokes equations of viscous heat-conductive gases in a bounded domain $\Omega \subset \mathbb{R}^2$ which describe the conservation of mass, momentum and total energy, and can be written in the following form:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$\rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla P = 0, \quad (1.2)$$

$$c_v \rho(\partial_t \theta + u \cdot \nabla \theta) - \kappa \Delta \theta + P \operatorname{div} u = \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\operatorname{div} u)^2. \quad (1.3)$$

Here we denote by ρ, θ and $u = (u_1, u_2)^t$ the density, temperature, and velocity, respectively. The physical constants μ, λ are the viscosity coefficients satisfying $\mu > 0, \lambda + \mu \geq 0, c_v > 0$ and $\kappa > 0$ are the specific heat at constant volume and thermal conductivity coefficient, respectively. P is the pressure which is a known function of ρ and θ , and in the case of an ideal gas P has the following form

$$P = R\rho\theta, \quad (1.4)$$

where $R > 0$ is a generic gas constant.

Let Ω be a periodic domain in \mathbb{R}^2 , or the unit square $[0, 1]^2$ in \mathbb{R}^2 . We will consider an initial boundary value problem for (1.1)–(1.3) in $Q := (0, \infty) \times \Omega$ with initial condition

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0) \quad \text{in } \Omega, \quad (1.5)$$

and boundary conditions:

$$\begin{aligned} (i) \quad & \rho, u, \theta \text{ are periodic in each } x_i \text{ for } 1 \leq i \leq 2, \text{ or} \\ (ii) \quad & u_i|_{x_i=0,1} = \partial_i u_j|_{x_i=0,1} = 0, \quad \forall 1 \leq i, j \leq 2, j \neq i, \\ & \partial_i \theta|_{x_i=0,1} = 0, \quad \forall 1 \leq i \leq 2. \end{aligned} \quad (1.6)$$

In the last decades significant progress has been made in the study of global in time existence for the system (1.1)–(1.6). With the assumption that the initial data are sufficiently small, Matsumura and Nishida [19, 20] first proved the global existence of smooth solutions to initial boundary value problems and the Cauchy problem for (1.1)–(1.3), and the existence of global weak solutions was shown by Hoff [12]. For large data, however, the global existence to (1.1)–(1.6) is still an open problem, except certain special cases, such as the spherically symmetric case in domains without the origin, see [15] for example. Recently, Feireisl [9, 10] obtained the global existence of the so-called “variational solutions” to (1.1)–(1.3) in the case of real gases in the sense that the energy equation is replaced by an energy inequality. However, this result excludes the case of ideal gases unfortunately. We mention that in the isentropic case, the existence of global weak solutions of the multidimensional compressible Navier-Stokes equations was first shown by Lions [18], and his result was then improved and generalized in [8] (also see [16, 17], and among others). Moreover, this kind of

weak solution with finite energy was shown to exist in $[0, \infty)$ as long as the density remains bounded in $L^\infty(\mathbb{T}^2)$ (cf. Desjardins [4]).

Xin [25], Rozanova [21] showed the non-existence of global smooth solutions when the initial density is compactly supported, or decreases to zero rapidly. Since the system (1.1)–(1.3) is a model of non-dilute fluids, these non-existence results are natural to expect when vacuum regions are present initially. Thus, it is very interesting to investigate whether a strong or smooth solution will still blow up in finite time, when there is no vacuum initially. Recently, Fan and Jiang [5] proved the following blow-up criteria for the local strong solutions to (1.1)–(1.6) in the case of two dimensions:

$$\lim_{T \rightarrow T^*} \left(\sup_{0 \leq t \leq T} \{ \|\rho\|_{L^\infty}, \|\rho^{-1}\|_{L^\infty}, \|\theta\|_{L^\infty} \}(t) + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla \rho\|_{L^2}^4 + \|u\|_{L^{r,\infty}}^{\frac{2r}{r-2}}) dt \right) = \infty,$$

or,

$$\lim_{T \rightarrow T^*} \left(\sup_{0 \leq t \leq T} \{ \|\rho\|_{L^\infty}, \|\rho^{-1}\|_{L^\infty}, \|\theta\|_{L^\infty} \}(t) + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla \rho\|_{L^2}^4) dt \right) = \infty,$$

provided $2\mu > \lambda$, where $T^* < \infty$ is the maximal time of existence of a strong solution (ρ, u) , $q_0 > 3$ is a certain number, $3 < r \leq \infty$ with $2/s + 3/r = 1$, and $L^{r,\infty} \equiv L^{r,\infty}(\Omega)$ is the Lorentz space.

In the isentropic case, the result in [5] reduces to

$$\lim_{T \rightarrow T^*} \left(\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla \rho\|_{L^2}^4) dt \right) = \infty, \quad (1.7)$$

provided $7\mu > 9\lambda$. Recently, Huang and Xin [14] established the following blow-up criterion in a 3-D smooth bounded domain, similar to the Beale-Kato-Majda criterion for ideal incompressible flows [1], for the isentropic compressible Navier-Stokes equations:

$$\lim_{T \rightarrow T^*} \int_0^T \|\nabla u\|_{L^\infty} dt = \infty, \quad (1.8)$$

provided

$$7\mu > \lambda. \quad (1.9)$$

Indeed, if the domain is a periodic or unit square domain in \mathbb{R}^2 , the blow-up criterion is refined by Fan, Jiang and Ni [6], to be

$$\lim_{T \rightarrow T^*} \sup_{0 \leq t \leq T} (\|\rho\|_{L^\infty} + \|\rho^{-1}\|_{L^\infty}) = \infty. \quad (1.10)$$

This result was recently improved by Sun, Wang and Zhang [22], in both two- and three-dimensional cases, to be

$$\lim_{T \rightarrow T^*} \sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} = \infty,$$

while a sharper criterion in the following form was given by Haspot in [11]:

$$\lim_{T \rightarrow T^*} \sup_{0 \leq t \leq T} \|\rho\|_{L^{(N+1+\epsilon)\gamma}} = \infty,$$

where N ($= 2, 3$) and γ are the spatial dimension and the specific heat ratio respectively, and ϵ is an arbitrary small number.

For the non-isentropic compressible Navier-Stokes equations, Fan, Jiang and Ou [7] established a blow-up criterion with additional upper bound of θ :

$$\lim_{T \rightarrow T^*} \left(\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \right) = \infty,$$

provided that the condition (1.9) is satisfied. This result reduces to the one in [14] in the isentropic regime.

The aim of this paper is to show that, the requirement of upper boundedness of θ in [7] can be removed and the condition (1.9) can be refined to be the usual physical condition $\lambda + \mu \geq 0$ for non-vacuum fluids in a 2-D domain. This result coincides the famous Beale-Kato-Majda criterion for ideal incompressible flows, and the criterion in [14] in the non-vacuum case. In contrast to [7], it is interesting to see here that the temperature θ allows to vanish in Ω , and more important, the temperature will not lead to the blow-up of strong solutions to the full Navier-Stokes equations. These are exactly the new points of this paper, in comparison with [7].

Moreover, it is interesting to see that the a priori assumption (2.1) is more concise than the one in [5, 7].

For the sake of generality, we will study the blow-up criterion for local strong solutions.

Before giving our main result, we state the following local existence of the strong solutions, the proof of which can be found in [3].

Proposition 1.1 (Local Existence) *Let Ω be a bounded domain in \mathbb{R}^2 as previously stated. Suppose that the initial data ρ_0, u_0, θ_0 satisfy*

$$\begin{aligned} \inf_{x \in \Omega} \rho_0(x) &> 0, \quad \rho_0 \in W^{1,q}(\Omega) \quad \text{for any } q > 2, \\ u_0 &\in H_0^1(\Omega) \cap H^2(\Omega), \quad \theta_0 \geq 0, \quad \theta_0 \in H^2(\Omega), \end{aligned} \tag{1.11}$$

and the compatibility conditions

$$\begin{aligned} \mu \Delta u_0 + (\mu + \lambda) \nabla \operatorname{div} u_0 - R \nabla(\rho_0 \theta_0) &= \rho_0^{1/2} g_1, \\ \kappa \Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + \nabla u_0^t|^2 + \lambda (\operatorname{div} u_0)^2 - R \rho_0 \theta_0 \operatorname{div} u_0 &= \rho_0^{1/2} g_2, \end{aligned} \tag{1.12}$$

for some $g_1, g_2 \in L^2(\Omega)$. Then there exist a positive constant T_0 and a unique strong solution (ρ, u, θ) to (1.1)–(1.6), such that

$$\begin{aligned} \rho &> 0, \quad \rho \in C([0, T_0]; W^{1,q}), \quad \rho_t \in C([0, T_0]; L^q), \\ u &\in C([0, T_0]; H_0^1 \cap H^2) \cap L^2(0, T_0; W^{2,q}), \\ u_t &\in L^\infty(0, T_0; L^2), \quad u_t \in L^2(0, T_0; H_0^1), \\ \theta &\geq 0, \quad \theta \in C([0, T_0]; H^2) \cap L^2(0, T_0; W^{2,q}), \\ \theta_t &\in L^\infty(0, T_0; L^2), \quad \theta_t \in L^2(0, T_0; H^1). \end{aligned} \tag{1.13}$$

By the regularity $(u_t, \theta_t) \in L^\infty(0, T_0; L^2)$, the quantities $\|u_t(T_0)\|_{L^2(\Omega)}$ and $\|\theta_t(T_0)\|_{L^2(\Omega)}$, redefined if necessary, are finite, which leads to the validity of the compatibility conditions at $t = T_0$. One may refer to *Remark 2* in [3] for the necessity of the compatibility conditions in (1.12).

Therefore, with the regularities in (1.13) and the new compatibility conditions at $t = T_0$, we are able to extend the solution to the time beyond T_0 . Now, we are interested in the question what happens to the solution if we extend the solution repeatedly. One possible case is that the solution exists in $[0, \infty)$, while another case is that the solution will blow-up in finite time in the sense of (1.13), that is, some of the regularities in (1.13) no longer hold.

Definition 1.1 $T^* \in (0, \infty)$ is called the maximal life-time of existence of a strong solution to (1.1)-(1.6) in the regularity class (1.13) if for any $0 < T < T^*$, (ρ, u, θ) solves (1.1)-(1.6) in $[0, T] \times \Omega$ and satisfies (1.13) with $T_0 = T$, and moreover, (1.13) does not hold for $T_0 = T^*$.

Now, we are ready to state the main theorem of this paper.

Theorem 1.1 (Blow-up Criterion) *Suppose that the assumptions in Proposition 1.1 are satisfied. Let (ρ, u, θ) be the strong solution obtained in Proposition 1.1. Then either this solution can be extended to $[0, \infty)$, or there exists a positive constant $T^* < \infty$, the maximal time of existence, such that the solution only exists in $[0, T]$ for every $T < T^*$, and*

$$\lim_{T \rightarrow T^*} \int_0^T \|\nabla u(t)\|_{L^\infty} dt = \infty.$$

□

We will prove Theorem 1.1 by contradiction in the next section. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|\nabla u\|_{L^1(0, T; L^\infty)}$ is bounded independent of any $T \in [0, T^*)$. The a priori estimates are then sufficient for us to apply the local existence theorem repeatedly to extend the local solution beyond the maximal time of existence T^* , consequently, contradicting the maximality of T^* .

The proof of this paper is based on the estimates for the effective viscous flux $(2\mu + \lambda)\operatorname{div} u - P$. We can obtain good estimates on the effective viscous flux at the first step of derivative estimates. This is the main ingredient of the estimates. It plays an important role in deriving other derivative estimates. This technique is applied in many previous situations for studying the Navier-Stokes equations (cf. [24, 6]), we adapt it here to establish the blow-up criteria for the full Navier-Stokes equations.

The rest of this paper is organized as follows. First, we will establish the estimates for all the zero-th order quantities of the solutions. Then, we will derive the crucial $\|u\|_{C([0, T], H^1(\Omega))}$ bound by utilizing the effective viscous flux, and then the estimates for other derivatives. Finally, we conclude the blow-up criteria by contradiction and continuity arguments.

Throughout this paper, we will use the following abbreviations:

$$\begin{aligned} L^p &\equiv L^p(\Omega), & H^m &\equiv H^m(\Omega), & H_0^m &\equiv H_0^m(\Omega), \\ L_t^p(X) &\equiv L^p(0, t; X(\Omega)), & C_t(X) &\equiv C([0, t], X(\Omega)). \end{aligned}$$

2 Proof of Theorem 1.1

Let $0 < T < T^*$ be arbitrary but fixed. Throughout this section, We denote by δ, ϵ various small positive constants, and moreover we denote by C (or C_X to emphasize the dependence of C on X) a general positive constant which may depend continuously on T^* .

Let (ρ, u, θ) be a strong solution to the problem (1.1)–(1.6) in the function space given in (1.13) on the time interval $[0, T]$. Suppose that $T^* < \infty$. We will prove Theorem 1.1 by a contradiction argument. To this end, we suppose that for any $T < T^*$,

$$\|\nabla u\|_{L^1(0,T;L^\infty)} \leq C < \infty, \quad (2.1)$$

we will deduce a contradiction to the maximality of T^* .

2.1 Zero-th order estimates

First, we show that the density ρ is bounded from below and above due to the assumption in (2.1). It is easy to see that the continuity equation (1.1) on the characteristic curve $\dot{\chi}(t) = u(\chi(t))$ can be written as

$$\frac{d}{dt}\rho(\chi(t), t) = -\rho(\chi(t), t)\operatorname{div} u(\chi(t), t).$$

Thus, by Gronwall's inequality and (2.1), one obtains that for any $x \in \bar{\Omega}$ and $t \in [0, T]$,

$$C^{-1} \leq \underline{\rho} \exp(-\|\operatorname{div} u\|_{L_T^1(L^\infty)}) \leq \rho(x, t) \leq \bar{\rho} \exp(\|\operatorname{div} u\|_{L_T^1(L^\infty)}) \leq C, \quad (2.2)$$

where $0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}$.

Multiplying (1.2) by u and summarizing the result by (1.3), we integrate to get

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |u|^2 + c_V \rho \theta \right) dx = 0. \quad (2.3)$$

Next, we show that θ is non-negative in $[0, T] \times \Omega$ (see also [9, 7]). Let $H(\theta)(x, t) := c_V \min\{-\theta(x, t), 0\}$. Clearly, $H'(\theta) \leq 0$ and $H''(\theta) = 0$. We multiply (1.3) by $H'(\theta)$ and integrate over Ω to obtain

$$\begin{aligned} & \int_{\Omega} \left(\rho(H(\theta)_t + u \cdot \nabla H(\theta)) + R\rho H(\theta) \operatorname{div} u \right) dx \\ &= \int_{\Omega} H'(\theta) \left(\kappa \Delta \theta + \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\operatorname{div} u)^2 \right) dx \\ &\leq \kappa \int_{\partial\Omega} H'(\theta) \frac{\partial \theta}{\partial n} dS - \kappa \int_{\Omega} H''(\theta) |\nabla \theta|^2 dx \\ &\leq 0 \end{aligned}$$

By the continuity equation (1.1), we integrate by parts to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho H(\theta) dx &\leq C \int_{\Omega} |\operatorname{div} u| |\rho H(\theta)| dx \\ &\leq -\|\operatorname{div} u\|_{L^\infty} \int_{\Omega} \rho H(\theta) dx. \end{aligned}$$

Utilizing (2.1) and Gronwall's inequality, we have

$$\int_{\Omega} \rho H(\theta) dx \equiv 0, \quad \forall t \in [0, T],$$

since $\theta_0 \geq 0$. Thus $\theta \geq 0$ by the definition of $H(\theta)$.

From (2.2), (2.3) and the non-negativeness of θ , we have

$$\sup_{0 \leq t \leq T} (\|u(t)\|_{L^2} + \|\theta(t)\|_{L^1}) \leq C, \quad \forall t \in [0, T]. \quad (2.4)$$

2.2 Estimates for derivatives

By multiplying (1.2) by u and integrating by parts, it follows immediately

$$\begin{aligned} \|\nabla u\|_{L_t^2(L^2)}^2 &\leq \frac{1}{2} \int_{\Omega} \rho |u|^2 dx(t) + \frac{1}{2} \int_{\Omega} \rho_0 |u_0|^2 dx + \int_0^t \int_{\Omega} |P| |\operatorname{div} u| dx ds \\ &\leq C + R \|\rho\|_{L_t^\infty(L^\infty)} \|\theta\|_{L_t^\infty(L^1)} \|\operatorname{div} u\|_{L_t^1(L^\infty)} \leq C, \quad \forall t \in [0, T]. \end{aligned} \quad (2.5)$$

Now, we are ready to control $\|u\|_{L^\infty(0,t;H^1)}$ and $\|u\|_{L^2(0,t;H^2)}$ by estimating the effective viscous flux $(2\mu + \lambda)\operatorname{div} u - P$, which is similar to the strategies in [24] and [6]. These are the key estimates in our proof. To simplify the statement, we denote by

$$V := -\operatorname{curl} u = \partial_2 u_1 - \partial_1 u_2$$

the vorticity, and by

$$F := (2\mu + \lambda)\operatorname{div} u - P$$

the effective viscous flux.

Lemma 2.1 (Key estimates) *Under the assumption in (2.1), we have for any $T < T^*$ that*

$$\sup_{0 \leq t \leq T} \|(\theta, \nabla u)(t)\|_{L^2}^2 + \int_0^T \|(\nabla \theta, \nabla F)\|_{L^2}^2 dt \leq C. \quad (2.6)$$

Proof. We first derive the following system from (1.1)-(1.2) and (1.4)-(1.6):

$$V_t + u \cdot \nabla V + V \operatorname{div} u - \partial_2 \left(\frac{1}{\rho} (\partial_2 V + \partial_1 F) \right) + \partial_1 \left(\frac{1}{\rho} (\partial_2 F - \partial_1 V) \right) = 0, \quad (2.7)$$

$$\begin{aligned} F_t + u \cdot \nabla F - (2\mu + \lambda) \left(\partial_1 \left(\frac{1}{\rho} (\partial_2 V + \partial_1 F) \right) + \partial_2 \left(\frac{1}{\rho} (\partial_2 F - \partial_1 V) \right) \right) \\ = O(1) (\partial_i u_j \partial_k u_l) - \gamma P \operatorname{div} u - (\gamma - 1) \Delta \theta, \end{aligned} \quad (2.8)$$

$$V|_{t=0} = \operatorname{curl} u_0, \quad F|_{t=0} = (2\mu + \lambda) \operatorname{div} u_0 - R \rho_0 \theta_0, \quad (2.9)$$

$$V|_{x_i=0,1} = \partial_i F|_{x_i=0,1} = 0, \quad i = 1, 2, \quad (2.10)$$

where $\gamma = 1 + R/c_V$. We multiply (2.7), (2.8) by V , $F/(2\mu + \lambda)$ respectively, and integrate by parts to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} (V^2 + \frac{F^2}{2\mu + \lambda}) dx + \int_{\Omega} \frac{(\partial_2 V + \partial_1 F)^2 + (\partial_2 F - \partial_1 V)^2}{\rho} dx \\
&= \int_{\Omega} \frac{1}{2} \operatorname{div} u \left(\frac{F^2}{2\mu + \lambda} - V^2 \right) - \frac{\gamma}{2\mu + \lambda} \int_{\Omega} F P \operatorname{div} u dx \\
&\quad + O(1) \int_{\Omega} \partial_i u_j \partial_k u_l F dx - \frac{\gamma - 1}{2\mu + \lambda} \int_{\Omega} F \Delta \theta dx \\
&=: \sum_{i=1}^4 I_i.
\end{aligned} \tag{2.11}$$

Denote the second integral on the left-hand side by I_5 . Then by integrating by parts and applying (2.2), we have

$$\begin{aligned}
I_5 &\geq C_0 \int_{\Omega} [(\partial_2 V + \partial_1 F)^2 + (\partial_2 F - \partial_1 V)^2] dx \\
&= C_0 (\|\nabla V\|_{L^2}^2 + \|\nabla F\|_{L^2}^2).
\end{aligned} \tag{2.12}$$

Noting that ρ is bounded from above, we obtain

$$|I_1 + I_2| \leq C \|\operatorname{div} u\|_{L^\infty} (\|V\|_{L^2}^2 + \|F\|_{L^2}^2 + \|\theta\|_{L^2}^2). \tag{2.13}$$

To bound I_3 , we need the following lemma.

Lemma 2.2 *For any $u \in H^1(\Omega)$ satisfying the boundary conditions in (1.6), we have*

$$\|\nabla u\|_{L^p} \leq C (\|\operatorname{div} u\|_{L^p} + \|\operatorname{curl} u\|_{L^p} + \|u\|_{L^2}), \quad \forall 2 \leq p < \infty.$$

The previous version of this lemma (cf. [2]) holds in case of a smooth domain Ω and $u \cdot n|_{\partial\Omega} = 0$. However, the conclusion can be easily adapted to our case. We can slightly modify the original proof by an extension argument, since the angles at corner points of our domain here are right-angles.

Noting also that

$$\operatorname{div} u = (F + P)/(2\mu + \lambda),$$

we have

$$\begin{aligned}
|I_3| &\leq C \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \|F\|_{L^2} \\
&\leq C \|\nabla u\|_{L^\infty} (\|V\|_{L^2} + \|F\|_{L^2} + \|\theta\|_{L^2} + \|u\|_{L^2}) \|F\|_{L^2} \\
&\leq C \|\nabla u\|_{L^\infty} (\|V\|_{L^2}^2 + \|F\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|u\|_{L^2}^2).
\end{aligned}$$

Finally, we get by integration by parts that

$$|I_4| \leq C \int_{\Omega} |\nabla \theta| |\nabla F| dx \leq \frac{C_0}{2} \|\nabla F\|_{L^2}^2 + C_1 \|\nabla \theta\|_{L^2}^2.$$

On the other hand, we can derive from (1.3) that

$$\begin{aligned} \frac{c_V}{2} \frac{d}{dt} \int_{\Omega} \rho \theta^2 dx + \int_{\Omega} |\nabla \theta|^2 dx &\leq C(\|\operatorname{div} u\|_{L^\infty} \|\theta\|_{L^2}^2 + \|\theta\|_{L^2} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}) \\ &\leq C\|\nabla u\|_{L^\infty} (\|V\|_{L^2}^2 + \|F\|_{L^2}^2 + \|\theta\|_{L^2}^2). \end{aligned} \quad (2.14)$$

Collecting the above estimates, we can draw the conclusion by applying Gronwall's inequality to (2.11) and (2.14). \square

Lemma 2.3 *With the assumption in (2.1), we have for any $0 \leq t \leq T$ that*

$$\|\nabla \rho(t)\|_{L^2} + \|u\|_{L_t^2(H^2)} + \|u_t\|_{L_t^2(L^2)} \leq C(1 + \|\theta\|_{L_t^2(H^2)}^{4\epsilon}).$$

Proof. Since u is a solution of the strictly elliptic system

$$-\mu \Delta u = f$$

where $f := -\rho u_t - \rho u \cdot \nabla u - \nabla F$, it follows from the classical regularity theory and the interpolation inequality that

$$\|u\|_{H^2} \leq C \left(\|u_t\|_{L^2} + \|u\|_{H^1}^{\frac{3}{2}} \|\nabla u\|_{H^2}^{\frac{1}{2}} + \|\theta\|_{L^\infty} \|\nabla \rho\|_{L^2} + \|\nabla \theta\|_{L^2} \right), \quad (2.15)$$

whence,

$$\|u\|_{L_t^1(H^2)} \leq C(1 + \|u_t\|_{L_t^2(L^2)} + \int_0^t \|\theta\|_{L^\infty} \|\nabla \rho\|_{L^2} ds), \quad \forall t \in [0, T]. \quad (2.16)$$

On the other hand, we would like to estimate $\|u_t\|_{L^2(0,t;L^2)}$ in terms of $\|u\|_{L^2(0,t;H^2)}$ to close the estimates. Taking the inner product of (1.2) with u_t in $L^2((0,t) \times \Omega)$, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\mu}{2} |D(u)|^2 + \frac{\lambda}{2} (\operatorname{div} u)^2 - P \operatorname{div} u \right) dx + \int_{\Omega} \rho u_t^2 dx \\ \leq C \int_{\Omega} |u| |\nabla u| |u_t| dx + \int_{\Omega} P_t \operatorname{div} u dx := J_1 + J_2, \end{aligned} \quad (2.17)$$

where $D(u) := (\nabla u + \nabla u_t)/2$. We calculate J_1 and J_2 as follows. By the interpolation inequality again, we get

$$\begin{aligned} \int_0^t J_1 ds &\leq C \int_0^t \|u\|_{H^1}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|u_t\|_{L^2} ds \\ &\leq \frac{1}{2} \int_0^t \int_{\Omega} \rho u_t^2 dx ds + C \|u\|_{L_t^2(H^2)}. \end{aligned}$$

From (1.1) and (1.3), we have

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u - \kappa \Delta \theta = 2\mu |D(u)|^2 + \lambda (\operatorname{div} u)^2, \quad (2.18)$$

thus, by virtue of integration by parts,

$$\begin{aligned} \left| \int_0^t J_2 ds \right| &\leq C \int_0^t \int_{\Omega} (P(|\nabla u|^2 + |u| |\nabla^2 u|) + |\nabla \theta| |\nabla \operatorname{div} u| + |\nabla u|^3) dx ds \\ &\leq C \|\theta\|_{L_t^\infty(L^2)} \|\nabla u\|_{L_t^1(L^\infty)} \|\nabla u\|_{L_t^\infty(L^2)} \\ &\quad + C \|\theta\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|\theta\|_{L_t^2(H^1)}^{\frac{1}{2}} \|u\|_{L_t^2(H^2)} \|u\|_{L_t^\infty(H^1)} \\ &\quad + C \|\nabla \theta\|_{L_t^2(L^2)} \|u\|_{L_t^2(H^2)} + C \|\nabla u\|_{L_t^1(L^\infty)} \|\nabla u\|_{L_t^\infty(L^2)}^2 \\ &\leq C + C \|u\|_{L_t^2(H^2)}. \end{aligned}$$

Note that $\int_{\Omega} P \operatorname{div} u dx(t) \leq C \|\theta\|_{C_t(L^2)} \|\nabla u\|_{C_t(L^2)} \leq C$, by the previous estimates. Thus, we conclude

$$\|u_t\|_{L_t^2(L^2)} \leq C(1 + \|u\|_{L_t^2(H^2)})^{\frac{1}{2}}, \quad \forall t \in [0, T]. \quad (2.19)$$

Next, we apply ∇ to the equation (1.1), then multiply the resulting equation by $\nabla \rho$ and integrate over Ω to get

$$\frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 dx \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^2}^2 + C \|u\|_{H^2} \|\nabla \rho\|_{L^2},$$

which gives, by Gronwall's inequality,

$$\|\rho(t)\|_{L^2} \leq C(1 + \|u\|_{L_t^1(H^2)}). \quad (2.20)$$

Substituting (2.16) and (2.19) into the above inequality and applying the integro-type Gronwall inequality, we have

$$\|\rho(t)\|_{L^2} \leq C(1 + \|u\|_{L_t^{\frac{1}{2}}(H^2)}). \quad (2.21)$$

From (2.15) again, we derive

$$\begin{aligned} \|u\|_{L_t^2(H^2)} &\leq C(1 + \|u_t\|_{L_t^2(L^2)} + \|\theta\|_{L_t^2(L^\infty)} \|\nabla \rho\|_{L_t^\infty(L^2)}) \\ &\leq C + C \|\theta\|_{L_t^2(H^1)}^{2(1-\epsilon)} \|\theta\|_{L_t^2(H^2)}^{2\epsilon} (1 + \|u\|_{L_t^{\frac{1}{2}}(H^2)}), \quad \forall t \in [0, T], \end{aligned} \quad (2.22)$$

which gives this lemma immediately. \square

Next, we derive further estimates for the derivatives of θ to close the above estimates.

Lemma 2.4 *Assuming (2.1), we have for any $T < T^*$ that*

$$\sup_{0 \leq t \leq T} \|\nabla \theta(t)\|_{L^2}^2 + \int_0^T (\|\theta\|_{H^2}^2 + \|\theta_t\|_{L^2}^2) dt \leq C.$$

Proof. Multiplying (1.3) by θ_t and integrating over Ω , we get

$$\begin{aligned}
& \frac{\kappa}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta|^2 dx + c_v \int_{\Omega} \rho \theta_t^2 dx \\
& \leq C \int_{\Omega} (|\rho| |u| |\nabla \theta| + |\rho| |\theta| |\operatorname{div} u| + |\nabla u|^2) |\theta_t| dx \\
& \leq C (\|u\|_{H^1} \|\theta\|_{H^1}^{1-\epsilon} \|\theta\|_{H^2}^{\epsilon} + \|\theta\|_{L^\infty} \|\operatorname{div} u\|_{L^2} + \|\nabla u\|_{L^4}^2) \|\theta_t\|_{L^2} \\
& \leq \delta \|\theta_t\|_{L^2}^2 + C_\delta \|u\|_{H^1}^2 (\|\theta\|_{H^1}^{2-2\epsilon} \|\theta\|_{H^2}^{2\epsilon} + \|u\|_{H^2}^2).
\end{aligned} \tag{2.23}$$

Since ρ is bounded from below, we choose δ small enough and then apply Gronwall's inequality to conclude

$$\|\nabla \theta(t)\|_{L^2}^2 + \|\theta_t\|_{L_t^2(L^2)}^2 \leq C(1 + \|\theta\|_{L_t^2(H^2)}^{4\epsilon}), \quad \forall 0 \leq t \leq T.$$

Again, we apply the elliptic regularity theory to derive

$$\begin{aligned}
\|\theta\|_{L_t^2(H^2)} & \leq C(\|\theta_t\|_{L_t^2(L^2)} + \|u \cdot \nabla \theta\|_{L_t^2(L^2)} + \|\theta \operatorname{div} u\|_{L_t^2(L^2)} + \|\nabla u\|_{L_t^2(L^4)}^2 + \|\theta\|_{L_t^2(H^1)}) \\
& \leq C(\|\theta_t\|_{L_t^2(L^2)} + \|u\|_{L_t^2(H^2)} \|\theta\|_{L_t^\infty(H^1)} + \|u\|_{L_t^\infty(H^1)} \|u\|_{L_t^2(H^2)} + 1) \\
& \leq C(\|\theta_t\|_{L_t^2(L^2)} + \|\theta\|_{L_t^\infty(H^1)} + 1).
\end{aligned}$$

Thus we can show this lemma easily from the above two inequalities by choosing $\epsilon < 1/4$. \square

Next, we will exploit the a priori estimates obtained so far to derive bounds on temporal derivatives and high-order derivatives.

Lemma 2.5 *Let (2.1) hold. For any $t \in [0, T]$, we have*

$$\sup_{0 \leq t \leq T} \left(\|(u_t(t), \theta_t(t))\|_{L^2}^2 + \|(u(t), \theta(t))\|_{H^2}^2 \right) + \int_0^T \|(u_t, \theta_t)\|_{H^1}^2 ds \leq C.$$

Proof. Now, taking ∂_t to the equation (1.2), multiplying then the resulting equation by u_t in $L^2(\Omega)$, integrating by parts, and employing (1.1) and (2.6), we find

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho u_t^2 dx + \int_{\Omega} \left(2\mu |D(u_t)|^2 + \lambda (\operatorname{div} u_t)^2 \right) dx \\
& = \int_{\Omega} P_t \operatorname{div} u_t dx - \int_{\Omega} \rho u \cdot \nabla [(u_t + u \cdot \nabla u) u_t] dx - \int_{\Omega} \rho u_t \cdot \nabla u \cdot u_t dx \\
& := K_1 + K_2 + K_3.
\end{aligned} \tag{2.24}$$

Observing that $P_t = R\rho_t\theta + R\rho\theta_t$ and $\rho_t = -\operatorname{div}(\rho u)$, and applying the interpolation inequality in two dimensions, we deduce

$$\begin{aligned}
|K_1| & \leq \delta \|\nabla u_t\|_{L^2}^2 + C_\delta \left(\|\theta_t\|_{L^2}^2 + \|\theta\|_{L^\infty}^2 (\|u\|_{L^\infty}^2 \|\nabla \rho\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2) \right) \\
& \leq \delta \|\nabla u_t\|_{L^2}^2 + C_\delta \left(\|\theta_t\|_{L^2}^2 + \|\theta\|_{H^1}^{2-\epsilon} \|\theta\|_{H^2}^{\epsilon} (\|u\|_{H^1}^{2-\epsilon} \|u\|_{H^2}^{\epsilon} + 1) \right),
\end{aligned}$$

for any $0 < \epsilon < \frac{1}{2}$, which follows

$$\begin{aligned} \int_0^t |K_1| ds &\leq \delta \|\nabla u_t\|_{L_t^2(L^2)}^2 + C_\delta \left(1 + \int_0^t \|\theta\|_{H^2}^\epsilon (\|u\|_{H^2}^\epsilon + 1) ds\right) \\ &\leq \delta \|\nabla u_t\|_{L_t^2(L^2)}^2 + C_\delta, \end{aligned} \quad (2.25)$$

by applying Lemmas 2.3 and 2.4.

Next,

$$\begin{aligned} \int_0^t |K_2| ds &\leq \int_0^t \int_\Omega \rho |u| (|u_t| |\nabla u_t| + |\nabla u|^2 |u_t| + |u| |\nabla^2 u| |u_t| + |\nabla u| |\nabla u_t|) dx ds \\ &\leq \int_0^t \|u\|_{H^1} (\|u_t\|_{L^3} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|u_t\|_{H^1} \\ &\quad + \|u\|_{H^1} \|u\|_{H^2} \|u_t\|_{H^1} + \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2}) ds \\ &\leq \int_0^t (\|u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{H^1}^{\frac{3}{2}} + \|u\|_{H^2} \|u_t\|_{H^1}) ds \\ &\leq \delta \|u_t\|_{L_t^2(H^1)}^2 + C_\delta, \end{aligned}$$

and

$$\int_0^t |K_3| ds \leq \int_0^t \|\nabla u\|_{L^2} \|u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{H^1}^{\frac{3}{2}} ds \leq \delta \|u_t\|_{L_t^2(H^1)}^2 + C_\delta.$$

Note that by the boundary conditions (1.6) and the constraint on viscosity coefficients μ, λ , we have (see also [18], pp.76)

$$\begin{aligned} \int_\Omega \left(2\mu |D(u_t)|^2 + \lambda (\operatorname{div} u_t)^2\right) dx &\geq 2\mu \int_\Omega \left(|D(u_t)|^2 - \frac{1}{N} (\operatorname{div} u_t)^2\right) dx \\ &\geq \bar{\mu} \|u_t\|_{H^1}^2 \end{aligned} \quad (2.26)$$

for some constant $\bar{\mu} > 0$. Then we conclude from (2.24), (2.26) and the estimates for K_1 through K_3 that

$$\sup_{0 \leq t \leq T} \|u_t(t)\|_{L^2}^2 + \|u_t\|_{L_t^2(H^1)}^2 \leq C. \quad (2.27)$$

From (1.2) and the elliptic regularity theory, we have

$$\begin{aligned} \|u\|_{L_t^\infty(H^2)} &\leq C (\|u_t\|_{L_T^\infty(L^2)} + \|u\|_{L_t^\infty(H^1)} \|\nabla u\|_{L_t^\infty(L^3)} \\ &\quad + \|\nabla \theta\|_{L_T^\infty(L^2)} + \|\theta\|_{L_t^\infty(L^\infty)} \|\nabla \rho\|_{L_t^\infty(L^2)}) \\ &\leq C (1 + \|u\|_{L_t^\infty(H^2)}^{\frac{1}{2}} + \|\theta\|_{L_t^\infty(H^2)}^\epsilon), \end{aligned} \quad (2.28)$$

for any $\epsilon > 0$. Similarly, we derive from (1.3) that

$$\|\theta\|_{L_t^\infty(H^2)} \leq C (\|\theta_t\|_{L_t^\infty(L^2)} + \|u\|_{L_t^\infty(H^2)}). \quad (2.29)$$

Substituting (2.28) into (2.29) and choosing $\epsilon < 1$, we obtain

$$\|u\|_{L_t^\infty(H^2)} + \|\theta\|_{L_t^\infty(H^2)} \leq C(1 + \|\theta_t\|_{L_t^\infty(L^2)}). \quad (2.30)$$

Next, we derive bounds for θ_t to close the desired energy estimates. Taking ∂_t on both sides of the equation (1.3), then multiplying the resulting equation by θ_t in $L^2(\Omega)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho \theta_t^2 dx + \kappa \int_{\Omega} |\nabla \theta_t|^2 dx \\ &= \int_{\Omega} R \rho \theta_t^2 \operatorname{div} u dx + \int_{\Omega} R \rho_t \theta \operatorname{div} u \theta_t dx + \int_{\Omega} R \rho \theta \operatorname{div} u_t \theta_t dx \\ &+ \int_{\Omega} [\mu(\nabla u + \nabla u^t) : (\nabla u_t + \nabla u_t^t) + 2\lambda \operatorname{div} u \operatorname{div} u_t] \theta_t dx \\ &- \int_{\Omega} \rho_t u \cdot \nabla \theta \theta_t dx - \int_{\Omega} \rho u_t \cdot \nabla \theta \theta_t dx - \int_{\Omega} \rho_t \theta_t^2 dx \\ &=: \sum_{i=1}^7 L_i. \end{aligned} \quad (2.31)$$

We have to estimate each term on the right-hand side of (2.31). From (1.1) and Sobolev's embedding theorem, we get

$$\begin{aligned} \int_0^t |L_1| ds &\leq C \int_0^t \|\theta_t\|_{H^1} \|\theta_t\|_{L^2} \|\operatorname{div} u\|_{L^3} ds \\ &\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_\delta \|u\|_{L_t^\infty(H^1)} \|u\|_{L_t^\infty(H^2)} \|\theta_t\|_{L_t^2(L^2)}^2 \\ &\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_\delta \|u\|_{L_t^\infty(H^2)}, \\ \int_0^t |L_2| ds &\leq \left| \int_{\Omega} R(\rho \operatorname{div} u + \nabla \rho \cdot u) \theta \operatorname{div} u \theta_t dx \right| \\ &\leq (\|\operatorname{div} u\|_{L_t^\infty(L^2)} + \|u\|_{L_t^\infty(H^1)} \|\nabla \rho\|_{L_t^\infty(L^2)}) \\ &\quad \times \|\theta\|_{L_t^\infty(H^1)} \|\operatorname{div} u\|_{L_t^2(H^1)} \|\theta_t\|_{L_t^2(H^1)} \\ &\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_\delta, \\ \int_0^t |L_3| ds &\leq C \int_0^t \|\theta_t\|_{H^1} \|\theta\|_{H^1} \|\operatorname{div} u_t\|_{L^2} ds \\ &\leq \delta \|\theta_t\|_{H^1}^2 + C_\delta, \\ \int_0^t |L_4| ds &\leq C \int_0^t \|\nabla u\|_{L^3} \|\nabla u_t\|_{L^2} \|\theta_t\|_{H^1} ds \\ &\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_\delta \|u\|_{L_t^\infty(H^1)} \|u\|_{L_t^\infty(H^2)} \|\nabla u_t\|_{L_t^2(L^2)}^2 \\ &\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_\delta \|u\|_{L_t^\infty(H^2)}, \end{aligned}$$

$$\begin{aligned}
\int_0^t |L_5| ds &\leq C \int_0^t \int_{\Omega} (\rho |\operatorname{div} u| + |u| |\nabla \rho|) |u| |\nabla \theta| |\theta_t| dx ds \\
&\leq C \int_0^t (\|\operatorname{div} u\|_{L^2} + \|\nabla \rho\|_{L^2} \|u\|_{H^1}) \|u\|_{H^1} \|\nabla \theta\|_{H^1} \|\theta_t\|_{H^1} ds \\
&\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_{\delta} \|\theta\|_{L_t^2(H^2)}^2 ds \\
&\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_{\delta}, \\
\int_0^t |L_6| ds &\leq C \int_0^t \|u_t\|_{H^1} \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^3} ds \\
&\leq \delta (\|u_t\|_{L_t^2(H^1)}^2 + \|\theta_t\|_{L_t^2(H^1)}^2) + C_{\delta} \|\theta_t\|_{L_t^2(L^2)}^2, \\
&\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_{\delta}, \\
\int_0^t |L_7| ds &\leq C \int_0^t (\|\operatorname{div} u\|_{L^2} + \|u\|_{H^1} \|\nabla \rho\|_{L^2}) \|\theta_t\|_{L^2} \|\theta_t\|_{H^1} ds \\
&\leq \delta \|\theta_t\|_{L_t^2(H^1)}^2 + C_{\delta}.
\end{aligned}$$

Now, we integrate (2.31) and utilize the estimates for L_1 through L_7 with δ sufficiently small to conclude

$$\|\theta_t(t)\|_{L^2}^2 + \|\theta_t\|_{L_t^2(H^1)}^2 \leq C(1 + \|u\|_{L_t^\infty(H^2)}), \quad 0 \leq t \leq T. \quad (2.32)$$

As a consequence of (2.28), (2.30) and (2.32), the current lemma is shown. \square

Finally, in the next lemma we show the additional L^q boundedness of the solution. The proof is exactly as in [7], however, we still reproduce it for the sake of completeness.

Lemma 2.6 *Let q be the same as in Theorem 1.1. Then,*

$$\sup_{0 \leq t \leq T} (\|\rho_t(t)\|_{L^q} + \|\rho(t)\|_{W^{1,q}}) \leq C, \quad (2.33)$$

$$\int_0^T (\|u(t)\|_{W^{2,q}}^2 + \|\theta(t)\|_{W^{2,q}}^2) dt \leq C. \quad (2.34)$$

Proof. Differentiating (1.1) with respect to x_j and multiplying the resulting equation by $|\partial_j \rho|^{q-2} \partial_j \rho$ in $L^2(\Omega)$, one deduces that

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} |\nabla \rho|^q dx &\leq C \int_{\Omega} (|\nabla u| |\nabla \rho|^q + |\rho| |\nabla \rho|^{q-1} |\nabla^2 u|) dx \\
&\leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^q}^q + C \|\nabla^2 u\|_{L^q} \|\nabla \rho\|_{L^q}^{q-1},
\end{aligned}$$

which gives

$$\begin{aligned}
\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^q} &\leq C \exp \left(\int_0^t \|\nabla u(s)\|_{L^\infty} ds \right) \left(\|\nabla \rho_0\|_{L^q} + \int_0^t \|\nabla^2 u(s)\|_{L^q} ds \right) \\
&\leq C(\sqrt{T})/\delta + \delta \|\nabla^2 u\|_{L_t^2(L^q)}
\end{aligned} \quad (2.35)$$

by Gronwall's inequality.

Using the regularity theory of elliptic equations again, we see that

$$\begin{aligned}\|u(t)\|_{W^{2,q}} &\leq C (\|u_t\|_{L^q} + \|u \cdot \nabla u\|_{L^q} + \|\nabla \rho\|_{L^q} + \|\nabla \theta\|_{L^q}) \\ &\leq C (\|\nabla u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^q} + \|\nabla \rho\|_{L^q} + \|\theta\|_{H^2}) \\ &\leq C (\|\nabla u_t\|_{L^2} + \|u\|_{H^2}^2 + \|\nabla \rho\|_{L^q} + \|\theta\|_{H^2}).\end{aligned}$$

If we integrate the above inequality over $(0, T)$ and make use of (2.35) as well as the estimates we have proved so far, we obtain

$$\int_0^T \|u(t)\|_{W^{2,q}}^2 dt \leq C, \quad (2.36)$$

and thus, from (2.35) one gets

$$\sup_{0 \leq t \leq T} \|\rho(t)\|_{W^{1,q}} \leq C.$$

Since $\rho_t = -u \nabla \rho - \rho \operatorname{div} u$, we also have

$$\|\rho_t(t)\|_{L^q} \leq \|u\|_{L^\infty} \|\nabla \rho\|_{L^q} + \|\rho\|_{L^\infty} \|\operatorname{div} u\|_{L^q} \leq C.$$

Then the boundedness of θ in $L^2(0, T; W^{2,q})$ follows from (1.3), (2.36) and the above inequality. The proof is finished. \square

2.3 Conclusions.

By virtue of all the above energy estimates, we obtain the bounds of the norms of (ρ, u, θ) in $[0, T] \times \Omega$ in the sense of (1.13) for any $T < T^*$. These bounds depend only on Ω , the initial data, and continuously on T^* (in fact, the bounds depend on T^* either polynomially or exponentially!). Thus, we can take $(\rho, u, \theta, \rho_t, u_t, \theta_t)|_{t=T}$, redefined if necessary, as the initial data at $t = T$ and apply Proposition 1.1 to extend the solution to $t = T + T_1$.

If $T + T_1 > T^*$, then it contradicts the maximality of T^* . Otherwise, we can continue to extend the solution by taking the values of the solution at $t = T + T_1$ as initial data again. Since the a priori estimates are independent of any $t < T^*$, the solution can be extended to $t = T + 2T_1$. Here, we remark that by applying Proposition 1.1, the solution can be extended from $t = T + T_1$ to $t = T + 2T_1$, since the local existence interval depends only on the initial data which, in our case, are bounded in any time interval $[0, \overline{T}]$ with a bound depending on \overline{T} only. Utilizing Proposition 1.1 repeatedly, there must exist a positive integer m , such that $T + mT_1 > T^*$. This also leads to the contradiction to the maximality of T^* . Therefore, the assumption (2.1) does not hold. This completes the proof of Theorem 1.1.

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